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LETTER TO THE EDITOR

# Gradient property of reduced bifurcation equation for systems with rotational symmetry

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**Abstract.** In this letter we shall study bifurcation theory in the presence of symmetry. It has been shown by Sattinger that when the symmetry group is the rotation group SO(3), the term of order 2 in the bifurcation equation is a gradient. We prove here that, in the SO(3) case, the third-order terms are gradient type for each representation $D^l$.

In this letter we shall study bifurcation theory in the presence of symmetry. We first give a very sketchy account of the main aspects of bifurcation theory of interest for the present work. This account is not meant to be an introduction to bifurcation theory (for that purpose we suggest Sattinger (1979) and Proc. CIMPA Spring School (1983).

Consider the system of differential equations

$$\dot{\boldsymbol{x}} = \boldsymbol{G}(\lambda, \boldsymbol{x}) \tag{1}$$

where $\lambda$ is a real parameter and $G$ is nonlinear in $x$, with $G(\lambda, 0) \equiv 0$. If we look for stationary solutions of (1), we immediately have one such solution, $x = 0$, for each value of $\lambda$. Local bifurcation theory permits us to determine the other 'small' solutions of (1) without effectively solving the system (1).

The implicit function theorem ensures that there is no solution near $(\lambda, 0)$ except in a neighbourhood (in the parameter space) of those points $\lambda_0$ in which $G_x(\lambda_0, 0)$ is not invertible, where $G_x(\lambda, 0)$ is the Fréchet derivative of $G(\lambda, x)$ at the zero solution. These points are called bifurcation points.

In a neighbourhood of a bifurcation point, these bifurcating solutions are given by the solutions of another equation, called the bifurcation equation, here indicated by $\boldsymbol{F}(\lambda, \boldsymbol{v}) = 0$, which acts on a considerably smaller space than the original equation. If $K$ and $R$ are the kernel and the range of $G_x(\lambda_0, 0)$, then $F: \mathbb{R} \times K \to R$, so that (except in very degenerate cases) we have to solve a finite-dimensional problem.

It is possible to write $F(\lambda, v)$ as $F(\lambda, v) = B_1(\lambda, v) + B_2(\lambda, v) + \ldots$;

$$F(\lambda, v) = \sum_m B_m(\lambda; v) \tag{2}$$

where each $B_m$ is $m$-linear in $v$.

Now consider the case in which problem (1) has a (linear) symmetry, i.e. there is a (linear) representation $\hat{T}_g$ of a group $G$ such that

$$\hat{T}_g G(\lambda, x) = G(\lambda, \hat{T}_g x). \tag{3}$$

In this case the same property holds for the bifurcation equation,

$$T_g F(\lambda, v) = F(\lambda, T_g v) \tag{4}$$

where $T_g$ is the restriction of $\hat{T}_g$ to the appropriate space.

The covariance property has therefore to be satisfied by each of the $B_m$'s, and this allows us to determine the general form of $F(\lambda, v)$: in fact, each $B_m$ can be considered as a tensor of $(K^{\otimes m} \otimes K^*)$, and the algebra of symmetric tensors on a vector space $V$ is isomorphous to the algebra of polynomials in the basis vectors of $V$.

So, in order to construct the more general bifurcation equation, it suffices to construct all the homogeneous covariant operators in $z_1, \ldots, z_n$ (basis vectors of $K$), and the condition of covariance becomes

$$(T_g F)(z_1, \ldots, z_n) = F(T_g z_1, \ldots, T_g z_n). \tag{5}$$

So, to construct the term of order $n$ of the bifurcation equation, it suffices to impose (5) for the generators of the group on operators homogeneous of degree $n$.

For a justification of the statements made above, and for a more detailed exposition of the theory, we refer to Sattinger (1978, 1979) and Proc. CIMPA Spring School (1983).

It was shown by Sattinger (1978, 1979) that when the symmetry group is the rotation group SO(3), the term of order two in the bifurcation equation is a gradient. This, apart from general theoretical interest, allows us to substitute—in a small neighbourhood of $(\lambda_0, 0)$, i.e. considering only the terms up to order two—the search for a solution of an $n$-dimensional algebraic system with the search for stationary points of a function on an $(n-1)$ unit sphere, the stability of the solutions being determined by the type of the corresponding stationary point.

We prove here that, in the SO(3) case, also the third-order terms are gradient type for each representation $D^l$. This allows us to use variational methods also in the case in which the quadratic term vanishes.

We denote covariant and scalar terms of order respectively 3 and 4 by $\psi^{(3)}_{LM;\lambda}$ and $\psi^{(4)}_{00;\lambda}$, where $M$ is an index which distinguishes the components of the vector $\psi^{(3)}_L$, $M = -L, \ldots, L$, and $\lambda$ is a parameter, corresponding to the total angular momentum of the coupling of the first two $D^l$'s.

Formulae for $\psi^{(4)}$ and $\psi^{(3)}$ are (Landau and Lifshitz 1958):

$$\psi^{(4)}_{00;\lambda} = \sum_{|m| \leq \lambda} \begin{pmatrix} \lambda & \lambda & 0 \\ m & -m & 0 \end{pmatrix} \psi^{(2)}_{\lambda m} \psi^{(2)}_{\lambda -m} \tag{6}$$

$$\psi^{(3)}_{lM;\lambda} = (-1)^{l-\lambda+M} (2l+1)^{1/2} \sum_{\mu+m=M} \begin{pmatrix} l & \lambda & -l \\ \mu & m & -M \end{pmatrix} \psi_{l\mu} \psi^{(2)}_{\lambda m} \tag{7}$$

$$\psi^{(2)}_{\lambda m} = (-1)^m (2\lambda+1)^{1/2} \sum_{\mu,\gamma} \begin{pmatrix} l & l & \lambda \\ \mu & \gamma & -m \end{pmatrix} \psi_{l\mu} \psi_{l\gamma}. \tag{8}$$

It is easy to check that all the $\psi^{(2)}_{\lambda m}$ (and therefore the $\psi^{(4)}_{00;\lambda}$ and $\psi^{(3)}_{lM,\lambda}$) with $\lambda$ odd are zero. In fact, using the symmetries of $3J$ symbols and the fact that $\lambda$ is odd,

$$\sum_{\mu\gamma} \begin{pmatrix} l & l & \lambda \\ \mu & \gamma & -m \end{pmatrix} \psi_\mu \psi_\gamma$$

$$= \sum_{\mu>\gamma} \begin{pmatrix} l & l & \lambda \\ \mu & \gamma & -m \end{pmatrix} \psi_\mu \psi_\gamma + \sum_{\mu<\gamma} \begin{pmatrix} l & l & \lambda \\ \mu & \gamma & -m \end{pmatrix} \psi_\mu \psi_\gamma$$

$$+\sum_{\mu}\begin{pmatrix} l & l & \lambda \\ \mu & \mu & -m \end{pmatrix}\psi_\mu\psi_\mu$$

$$=\sum_{\mu>\gamma}\begin{pmatrix} l & l & \lambda \\ \mu & \gamma & -m \end{pmatrix}\psi_\mu\psi_\gamma+(-1)^\lambda\sum_{\mu>\gamma}\begin{pmatrix} l & l & \lambda \\ \mu & \gamma & -m \end{pmatrix}\psi_\mu\psi_\gamma$$

$$+\sum_{\mu}\begin{pmatrix} l & l & \lambda \\ \mu & \mu & -m \end{pmatrix}\psi_\mu\psi_\mu$$

$$=\sum_{\mu}\begin{pmatrix} l & l & \lambda \\ \mu & \mu & -m \end{pmatrix}\psi_\mu\psi_\mu=(-1)^\lambda\sum_{\mu}\begin{pmatrix} l & l & \lambda \\ \mu & \mu & -m \end{pmatrix}\psi_\mu\psi_\mu=0.$$

It is also easy to check that $\lambda$ actually suffices to parametrise both $\psi^{(4)}$'s and $\psi^{(3)}$'s, since $D^l\otimes D^l=D^{2l}\oplus D^{2l-1}\oplus\ldots\oplus D^1\oplus D^0$.

Now, in order to show

$$\partial\psi^{(4)}_{00;\lambda}/\partial z_{-R}=\alpha\psi^{(3)}_{lR;\lambda} \tag{9}$$

(where, of course, $z_m\equiv\psi^{(1)}_{lm}$), we write $\psi^{(4)}$ as

$$\psi^{(4)}_{00;\lambda}=\sum_{\mu,\gamma,m}A(\lambda;m,\mu_1,\gamma_1)z_{\mu_1}z_{\mu_2}z_{\gamma_1}z_{\gamma_2}\qquad \mu_1+\mu_2=m=-\gamma_1-\gamma_2$$

and the derivative will be written in the form

$$\begin{aligned}\partial\psi^{(4)}_{00;\lambda}/\partial z_R=&\sum A(\lambda;m,\mu_1,\gamma_1)(\delta_{\mu_1R}+\delta_{\mu_2R})z_{m-R}z_{\gamma_1}z_{\gamma_2}\\&+A(\lambda;m,\mu_1,\gamma_1)(\delta_{\gamma_1R}+\delta_{\gamma_2R})z_{-m-R}z_{\mu_1}z_{\mu_2}.\end{aligned}$$

Now, let us consider the term $A(\lambda;m,\mu_1,\gamma_1)(\delta_{\mu_1,R}+\delta_{\mu_2,R})z_{m-R}z_{\gamma_1}z_{\gamma_2}$; this is the sum of two terms, the second of which is obtained from the first by

$$\begin{pmatrix} l & l & \lambda \\ \mu_1 & \mu_2 & m \end{pmatrix}\to\begin{pmatrix} l & l & \lambda \\ \mu_2 & \mu_1 & m \end{pmatrix}=(-1)^\lambda\begin{pmatrix} l & l & \lambda \\ \mu_1 & \mu_2 & m \end{pmatrix}.$$

So, for $\lambda$ even we have

$$\alpha_1\equiv\sum A(\lambda;m,\mu_1,\gamma_1)2\delta_{\mu_1,R}z_{m-R}z_{\gamma_1}z_{\gamma_2} \tag{10}$$

$$\alpha_1=2\sum_{|m|\leqslant\lambda}\frac{(-1)^{\lambda-m}}{(2\lambda+1)^{1/2}}\begin{pmatrix} l & l & \lambda \\ R & (m-R) & -m \end{pmatrix}\sum_{\substack{|\gamma_1|\leqslant l\\|m+\gamma_1|\leqslant l}}\begin{pmatrix} l & l & \lambda \\ \gamma_1 & (-m-\gamma_1) & m \end{pmatrix}z_{m-R}z_{\mu_1}z_{\mu_2}. \tag{11}$$

We can work out the second part the same way, defining

$$\alpha_2\equiv\sum A(\lambda;m,\mu_1,\gamma_1)(\delta_{\gamma_1R}+\delta_{\gamma_2R})z_{-m-R}z_{\gamma_1}z_{\gamma_2}.$$

It results in $\alpha_1=\alpha_2$, and therefore

$$\partial\psi^{(4)}_{00;\lambda}/\partial z_R=2\alpha_1 \tag{12}$$

with $\alpha_1$ given by (11).

As for $\psi^{(3)}$'s, we have, after a very short computation, combining (7) and (8),

$$\psi^{(3)}_{lM;\eta} = [(2l+1)(2\eta+1)]^{1/2} \sum_{|K|\leq\eta} (-1)$$
$$\times \begin{pmatrix} l & \eta & l \\ M+K & -K & -M \end{pmatrix} \sum_{\substack{|\varphi|\leq l \\ |K+\varphi|\leq l}} \begin{pmatrix} l & l & \eta \\ \varphi & (-K-\varphi) & K \end{pmatrix} z_{K+M} z_{\varphi} z_{-K-\varphi}.$$

Now, using the symmetries of $3J$ symbols, imposing $M=-R$, and using (12), we obtain (9) with $\alpha$ given by

$$\alpha = (-1)^{l+R} 4/(2l+1)^{1/2}. \tag{13}$$

After the completion of this work, D H Sattinger told us this same result was proved by L Michel (unpublished work), making use of different methods.

This work is the author's thesis for a doctor's degree. He could not have done it without Sunday.

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